

Find two linearly independent series solutions of $(2x - 1)y'' - xy' + 2y = 0$ about $x = 0$.

SCORE: ____ / 13 PTS

$$(2x - 1) \left[\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \right] - x \left[\sum_{n=1}^{\infty} na_n x^{n-1} \right] + 2 \left[\sum_{n=0}^{\infty} a_n x^n \right] = 0$$

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=1}^{\infty} 2(n+1)na_{n+1} x^n - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$-2a_2 + 2a_0 + \sum_{n=1}^{\infty} [2(n+1)na_{n+1} - (n+2)(n+1)a_{n+2} - (n-2)a_n] x^n = 0$$

$$-2a_2 + 2a_0 = 0 \Rightarrow a_2 = a_0$$

$$2(n+1)na_{n+1} - (n+2)(n+1)a_{n+2} - (n-2)a_n = 0 \Rightarrow a_{n+2} = \frac{2n(n+1)a_{n+1} - (n-2)a_n}{(n+2)(n+1)} \text{ for } n \in \mathbb{Z}^+$$

$$a_0 = 1, \quad a_1 = 0$$

$$a_2 = a_0 = 1$$

$$a_3 = \frac{2(1)(2)a_2 - (-1)a_1}{(3)(2)} = \frac{4}{(3)(2)} = \frac{2}{3}$$

$$a_4 = \frac{2(2)(3)a_3 - (0)a_2}{(4)(3)} = \frac{8}{(4)(3)} = \frac{2}{3}$$

$$y_1 = 1 + x^2 + \frac{2}{3}x^3 + \frac{2}{3}x^4 + \dots$$

②

$$a_0 = 0, \quad a_1 = 1$$

$$a_2 = 0$$

$$a_3 = \frac{2(1)(2)a_2 - (-1)a_1}{(3)(2)} = \frac{1}{(3)(2)} = \frac{1}{6}$$

$$a_4 = \frac{2(2)(3)a_3 - (0)a_2}{(4)(3)} = \frac{2}{(4)(3)} = \frac{1}{6}$$

$$a_5 = \frac{2(3)(4)a_4 - (1)a_3}{(5)(4)} = \frac{\frac{23}{6}}{(5)(4)} = \frac{23}{120}$$

$$y_1 = x + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{23}{120}x^5 + \dots$$

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① POINT EACH
EXCEPT
AS NOTED

Find two linearly independent series solutions of $2x^2 y'' + (2x^2 + 3x)y' - y = 0$ about $x = 0$.

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$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + (2x^2 + 3x) \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

① POINT EACH
EXCEPT
AS NOTED

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r+1} + \sum_{n=0}^{\infty} 3(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=1}^{\infty} 2(n+r-1)a_{n-1} x^{n+r} + \sum_{n=0}^{\infty} 3(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$2r(r-1)a_0 x^r + 3ra_0 x^r - a_0 x^r + \sum_{n=1}^{\infty} [(2(n+r)(n+r-1) + 3(n+r) - 1)a_n + 2(n+r-1)a_{n-1}] x^{n+r} = 0$$

$$(2r(r-1) + 3r - 1)a_0 x^r + \sum_{n=1}^{\infty} [(2(n+r)(n+r-1) + 3(n+r) - 1)a_n + 2(n+r-1)a_{n-1}] x^{n+r} = 0$$

$$2r(r-1) + 3r - 1 = 0 \Rightarrow 2r^2 + r - 1 = 0 \Rightarrow (2r-1)(r+1) = 0 \Rightarrow r = \frac{1}{2}, -1$$

$$(2(n+r)(n+r-1) + 3(n+r) - 1)a_n + 2(n+r-1)a_{n-1} = 0 \Rightarrow$$

$$a_n = -\frac{2(n+r-1)}{2(n+r)(n+r-1) + 3(n+r) - 1} a_{n-1} = -\frac{2(n+r-1)}{(2(n+r)-1)(n+r+1)} a_{n-1} = -\frac{2(n+r-1)}{(2n+2r-1)(n+r+1)} a_{n-1}$$

$$r = \frac{1}{2} \Rightarrow a_n = -\frac{2(n-\frac{1}{2})}{2n(n+\frac{3}{2})} a_{n-1} = -\frac{2n-1}{n(2n+3)} a_{n-1}$$

$$r = -1 \Rightarrow a_n = -\frac{2(n-2)}{(2n-3)n} a_{n-1}$$

$$a_0 = 1$$

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$$a_1 = -\frac{(1)}{(1)(5)} a_0 = -\frac{1}{5}$$

$$a_1 = -\frac{2(-1)}{(-1)(1)} a_0 = -2$$

$$a_2 = -\frac{(3)}{(2)(7)} a_1 = \frac{1 \cdot 3}{(1 \cdot 2)(5 \cdot 7)}$$

$$a_2 = -\frac{2(0)}{(1)(2)} a_1 = 0 = a_3 = a_4 = a_5 = \dots$$

$$a_3 = -\frac{(5)}{(3)(9)} a_2 = -\frac{1 \cdot 3 \cdot 5}{(1 \cdot 2 \cdot 3)(5 \cdot 7 \cdot 9)}$$

$$y_1 = x^{\frac{1}{2}} \left(1 - \frac{1}{5}x + \frac{1 \cdot 3}{(1 \cdot 2)(5 \cdot 7)}x^2 - \frac{1 \cdot 3 \cdot 5}{(1 \cdot 2 \cdot 3)(5 \cdot 7 \cdot 9)}x^3 + \dots \right)$$

$$y_2 = x^{-1} (1 - 2x) = \frac{1}{x} - 2$$

$$y_1 = x^{\frac{1}{2}} \left(1 - \frac{1}{5}x + \frac{3}{70}x^2 - \frac{1}{126}x^3 + \dots \right)$$

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